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A Unified Approach to Certain Counterexamples in Approximation Theory in Connection with a Uniform Boundedness Principle with Rates

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1. INTRODUCTION

Let X, Y be normed linear spaces with norms $\|\circ\|_X$, $\|\circ\|_Y$, respectively, and let $U \subset X$ be a linear subspace with seminorm $|\circ|_U$. Consider the intermediate spaces $U \subset X_\omega \subset X$,

$$X_\omega := \{f \in X; \mathscr{K}(t, f; X, U) = \mathscr{O}(\omega(t)), t \to 0+\}, \tag{1.1}$$

where the $\mathscr{K}$-functional is defined for $f \in X$, $t \geqslant 0$ by

$$\mathscr{K}(t, f) := \mathscr{K}(t, f; X, U) := \inf_{g \in U} \{\|f - g\|_X + t\,|g|_U\} \tag{1.2}$$

and ω is a modulus of continuity, namely a function, continuous and monotone increasing on $[0, \infty)$, satisfying the properties (cf. [17, p. 96 ff.])

$$\omega(0) = 0, \qquad \omega(t) > 0 \qquad \text{for} \quad t > 0, \tag{1.3}$$
$$\omega(t_1 + t_2) \leqslant \omega(t_1) + \omega(t_2).$$

Let $\mathbf{N}$ be the set of natural numbers and $\{R_n\}_{n \in \mathbf{N}}$ a sequence of operators on X into Y (e.g., remainders, cf. Section 3) which are sublinear and bounded, i.e., the operator norm

$$\|R_n\|_{[X,Y]} := \sup_{f \neq 0} \|R_n f\|_Y / \|f\|_X$$

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is finite for each $n \in \mathbf{N}$. If $\{\varphi_n\}_{n\in\mathbf{N}}$ is a sequence of real numbers satisfying

$$\varphi_n > 0 \text{ for } n \in \mathbf{N}, \qquad \lim_{n\to\infty} \varphi_n = 0 \text{ monotonely,} \tag{1.4}$$

then one has the familiar direct approximation (or Jackson-type) theorem (cf. [6] and the literature cited there):

THEOREM 1. *If the operators R_n satisfy the boundedness condition*

$$\|R_n f\|_Y \leqslant C \|f\|_X \qquad (f \in X, n \in \mathbf{N}) \tag{1.5}$$

and the Jackson-type inequality

$$\|R_n g\|_Y \leqslant C\varphi_n |g|_U \qquad (g \in U, n \in \mathbf{N}), \tag{1.6}$$

then for $f \in X_\omega$ one has the rate of convergence

$$\|R_n f\|_Y = \mathcal{O}(\omega(\varphi_n)) \qquad (n \to \infty). \tag{1.7}$$

Indeed, for any $f \in X$, $g \in U$

$$\|R_n f\|_Y \leqslant \|R_n(f-g)\|_Y + \|R_n g\|_Y \leqslant C\{\|f-g\|_X + \varphi_n |g|_U\},$$

and therefore in view of the definitions (1.1–1.2)

$$\|R_n f\|_Y \leqslant C\mathcal{K}(\varphi_n, f) = \mathcal{O}(\omega(\varphi_n))$$

for any $f \in X_\omega$.

It is the purpose of this note to show that the assertion of Theorem 1 is sharp. More specifically, for moduli of continuity satisfying

$$\sup_{t>0} \omega(t)/t = \infty \tag{1.8}$$

it will be shown in Section 2 (cf. Theorem 2, Corollary 1) that for rather general sequences $\{R_n\}$ (e.g., the operators R_n need not be commutative) there exists an element $f_\omega \in X_\omega$ for which the rate (1.7) cannot be improved to $o(\omega(\varphi_n))$. The method of proof essentially consists in the familiar gliding hump method (cf. [5, p. 18]), but now equipped with rates. Indeed, the results of Section 2 may also be considered as a contribution to the question of how to treat the classical uniform boundedness principle with rates (cf. Theorem 3). Another important feature is that Bernstein inequalities of type (2.1, 2.2, 2.11) are used in the course of the proof. In Section 3 some first applications are outlined emphazising the flexibility of this approach. For the interconnection with theorems of Banach–Steinhaus-type (cf. [3] and the literature cited there) we refer to [8].

2. The Gliding Hump Method with Rates

Let us commence with a general approach to certain counterexamples in approximation theory.

Theorem 2. *Let X be a Banach space, Y a normed linear space, $U \subset X$ a linear subspace, and X_ω be given by* (1.1). *Let $\{\varphi_n\}$ and ω satisfy* (1.4) *and* (1.3, 1.8), *respectively. If for a sequence of sublinear, bounded operators R_n of X into Y there exist elements $h_n \in U$ such that for all $n \in \mathbf{N}$*

$$\|h_n\|_X \leqslant C_1, \tag{2.1}$$

$$|h_n|_U \leqslant C_2 \varphi_n^{-1}, \tag{2.2}$$

$$0 < C_3 \leqslant \|R_n h_n\|_Y, \tag{2.3}$$

then for each space X_ω there exists an element $f_\omega \in X_\omega$ such that

$$\|R_n f_\omega\|_Y \neq o(\omega(\varphi_n)) \qquad (n \to \infty). \tag{2.4}$$

Proof. First of all we note that (1.3) always implies (cf. [17, p. 99])

$$\omega(s)/s \leqslant 2\omega(t)/t \qquad \text{for any} \quad s \geqslant t > 0. \tag{2.5}$$

Assume that for each $f \in X_\omega$

$$\|R_n f\|_Y = o(\omega(\varphi_n)) \quad (n \to \infty). \tag{2.6}$$

Starting with an arbitrary $n_1 \in \mathbf{N}$, one may successively construct a monotonely increasing subsequence $\{n_k\} \subset \mathbf{N}$ such that the following conditions are simultaneously satisfied ($k \geqslant 2$):

$$\omega(\varphi_{n_k}) \leqslant (1/2)\, \omega(\varphi_{n_{k-1}}), \tag{2.7}$$

$$\sum_{j=1}^{k-1} \omega(\varphi_{n_j})/\varphi_{n_j} \leqslant \omega(\varphi_{n_k})/\varphi_{n_k}, \tag{2.8}$$

$$\|R_{n_{k-1}}\|_{[X,Y]} \leqslant (C_3/6C_1)\, \omega(\varphi_{n_{k-1}})/(\omega(\varphi_{n_k}), \tag{2.9}$$

$$\|R_{n_k} g_{k-1}\|_Y \leqslant (C_3/3)\, \omega(\varphi_{n_k}), \tag{2.10}$$

$$g_{k-1} := \sum_{j=1}^{k-1} \omega(\varphi_{n_j})\, h_{n_j} \in U.$$

Indeed, (2.7, 2.9) may be satisfied in view of (1.3–1.4) and (2.8) in view of (1.8), (2.5), whereas (2.10) is a consequence of the assumption (2.6). By (2.1, 2.7) it follows that

$$\sum_{j=1}^{\infty} \|\omega(\varphi_{n_j})\, h_{n_j}\|_X \leqslant C_1 \sum_{j=1}^{\infty} \omega(\varphi_{n_j}) \leqslant C_1 \omega(\varphi_{n_1}) \sum_{j=1}^{\infty} 2^{-j+1} < \infty.$$

Therefore $g_\omega := \sum_{j=1}^{\infty} \omega(\varphi_{n_j})\, h_{n_j}$ is well defined as an element of X since X is complete. Moreover, $g_\omega \in X_\omega$. Indeed, for each $t \in (0, \varphi_{n_1})$ there exists $k_t := k \in \mathbf{N}$ such that $\varphi_{n_{k+1}} \leqslant t < \varphi_{n_k}$. Using the corresponding $g_k \in U$ and conditions (2.1–2.2), (2.7–2.8), and finally (1.3), (2.5), one has in view of definition (1.2)

$$\begin{aligned}
\mathscr{K}(t, g_\omega) &\leqslant \| g_\omega - g_k\|_X + t\, | g_k|_U \\
&= \left\| \sum_{j=k+1}^{\infty} \omega(\varphi_{n_j})\, h_{n_j} \right\|_X + t \left| \sum_{j=1}^{k} \omega(\varphi_{n_j})\, h_{n_j} \right|_U \\
&\leqslant 2C_1 \omega(\varphi_{n_{k+1}}) + 2C_2 t\omega(\varphi_{n_k})/\varphi_{n_k} \\
&\leqslant (2C_1 + 4C_2)\, \omega(t).
\end{aligned}$$

This proves that $g_\omega \in X_\omega$. Applying R_{n_k} to

$$g_\omega = \omega(\varphi_{n_k})\, h_{n_k} + g_{k-1} + (g_\omega - g_k),$$

one obtains by (2.3) and (2.9–2.10) that

$$\begin{aligned}
\|R_{n_k} g_\omega\|_Y &\geqslant \|R_{n_k}\omega(\varphi_{n_k})\, h_{n_k}\|_Y - \|R_{n_k} g_{k-1}\|_Y \\
&\quad - \|R_{n_k}\|_{[X,Y]}\, \| g_\omega - g_k\|_X \\
&\geqslant C_3 \omega(\varphi_{n_k})[1 - \tfrac{1}{3} - \tfrac{1}{3}].
\end{aligned}$$

Of course, this is a contradiction to (2.6), proving the assertion.

The preceding proof contains all the ingredients of the classical gliding hump method, including the construction of a convergent series such that the kth element of the series is "large with respect to the operator R_{n_k}." As a first contribution to the present treatment one may consider that of Teljakovskii [16] concerning multipliers of uniform convergence, reflecting, however, rather specific features of the one-dimensional trigonometric system. This was further developed in [13] in connection with multipliers of strong convergence for regular biorthogonal systems but still using a very special projection property.

Apart from the order $\omega(\varphi_{n_k})$ built in, the main distinction from the classical gliding hump method is that the limit g_ω of the series occurring in the course of the proof is not only an element of the underlying Banach space X but of the subspace X_ω, too. To this end condition (2.2) is needed which, in connection with (2.1), may be interpreted as a weak form of a Bernstein-type inequality for the sequence $\{h_n\}$. In fact, the standard version of such an inequality would read

$$|h_n|_U \leqslant C\varphi_n^{-1} \|h_n\|_X, \tag{2.11}$$

which together with (2.1) implies (2.2) (cf. treatment of the examples in Sections 3.1, 3.6). Naturally the use of a Bernstein-type inequality does not surprise one in view of the fact that Theorem 2 is a first step in a direction that is inverse to the Jackson-type Theorem 1 (cf. Corollary 1).

For a further interpretation of the conditions (2.1–2.3) consider first (2.1, 2.3). These conditions state that the operator norms $\|R_n\|$ are bounded away from zero, namely,

$$\|R_n\|_{[X,Y]} \geqslant \|R_n h_n\|_Y / \|h_n\|_X \geqslant C_3/C_1,$$

i.e., a condition of type (1.5) should be best possible. In the same sense conditions (2.2–2.3) state that

$$\|R_n h_n\|_Y \geqslant (C_3/C_2)\, \varphi_n\, |h_n|_U,$$

i.e., the Jackson-type inequality (1.6) should be best possible. In fact, in terms of the seminorm (cf. (1.1–1.2))

$$|f|_\omega := \sup_{t>0} \mathscr{K}(t, f)/\omega(t) \qquad (f \in X_\omega)$$

the assertions of Theorems 1, 2 may be summarized to the following result, stating that the estimate (1.7) cannot be improved.

COROLLARY 1. *Under the assumptions of Theorems* 1, 2, *including that conditions* (1.6), (2.2) *be satisfied for the same sequence* $\{\varphi_n\}$, *there exist constants* c_0, C_0 *such that*

$$0 < c_0 \leqslant \sup_{0<|f|_\omega<\infty} [\limsup_{n\to\infty} \|R_n f\|_Y / \omega(\varphi_n)] / |f|_\omega \leqslant C_0 < \infty.$$

Proof. The upper estimate is a consequence of the proof of Theorem 1 (with $C_0 = C$). Concerning the lower estimate, first observe that the Jackson-type inequality (1.6) together with (1.8), (2.5) now implies that (2.6) is not merely an assumption but actually satisfied for each $f \in U$. Therefore a subsequence may be constructed such that conditions (2.7–2.9) as well as

(2.10) hold true. This leads to a constructive version of the proof of Theorem 2, actually delivering some $g_\omega \in X_\omega$ such that

$$0 \neq |g_\omega|_\omega \leqslant 2C_1 + 4C_2, \qquad \|R_{n_k} g_\omega\|_Y \geqslant (C_3/3)\,\omega(\varphi_{n_k}).$$

Thus the lower estimate follows with $c_0 = C_3/6(C_1 + 2C_2)$.

The result of Theorem 2 may also be interpreted as a uniform boundedness principle with rates. Indeed,

THEOREM 3. *Let the assumptions of Theorem* 2 *be valid but with* (2.3) *replaced by*

$$\|R_n h_n\|_Y \geqslant C_3^* \|R_n\|_{[X,Y]} \qquad (n \in \mathbf{N}). \tag{2.3)*}$$

If for every $f \in X_\omega$

$$\|R_n f\|_Y = o(1) \qquad (n \to \infty), \tag{2.12}$$

then the operator norms necessarily satisfy the growth condition

$$\|R_n\|_{[X,Y]} = o(1/\omega(\varphi_n)) \qquad (n \to \infty). \tag{2.13}$$

Proof. Assume that (2.13) does not hold. Then for $\tilde{R}_n := \omega(\varphi_n) R_n$ one has that $\|\tilde{R}_n\|_{[X,Y]} \geqslant C$, at least for a subsequence. By (2.3)* it follows that the sequence $\{\tilde{R}_n\}$ satisfies (2.3) with $C_3 = C_3^* C$. Hence an application of Theorem 2 to $\{\tilde{R}_n\}$ delivers an element $f_\omega \in X_\omega$ such that $\|\tilde{R}_n f_\omega\|_Y \neq o(\omega(\varphi_n))$, a contradiction to (2.12).

Concerning condition (2.3)*, one may mention that the definition of the operator norm $\|R_n\|_{[X,Y]}$ actually implies the existence of some $f_n \in X$ (even $f_n \in U$ if U is dense in X) such that $\|f_n\|_X = 1$ and $\|R_n f_n\|_Y \geqslant c\,\|R_n\|_{[X,Y]}$, where $0 < c < 1$ is a given constant.

In present terms the classical uniform boundedness principle states that $\|R_n f\|_Y = o(1)$ for every f of the whole Banach space X implies $\|R_n\|_{[X,Y]} = \mathcal{O}(1)$ (and not $o(1)$). It reflects the fact that the limiting case $X_\omega = X$, i.e., $\omega(t) = \text{const.}$, is excluded by (1.3). In this connection we also note that the assertion of Theorem 3 cannot be true for the other limiting case $\omega(t) = t$; in fact it is excluded by (1.8) (cf. Section 3.4).

3. APPLICATIONS

The purpose of this section is to treat some first examples illustrating the usefulness of Theorem 2. Most of the results, possibly apart from those of Sections 3.4–3.6, are quite standard, contained in many textbooks. It is hoped, however, that the present treatment at least indicates the unified approach to the subject. For further applications, including numerical solutions of initial value problems, see [7, 8].

Concerning the applications to periodic problems given in Sections 3.1–3.4, let $X_{2\pi}$ be either $C_{2\pi}$ or $L^p_{2\pi}$, $1 \leqslant p < \infty$, the spaces of 2π-periodic continuous or Lebesgue-integrable functions f with finite norms

$$\|f\|_{X_{2\pi}} := \max_x |f(x)| \qquad \text{or} \qquad := \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(u)|^p \, du \right\}^{1/p},$$

respectively. If $X^{(r)}_{2\pi}$ denotes the subspace of functions r-times differentiable in $X_{2\pi}$ with seminorm $|f|_{X^{(r)}_{2\pi}} := \|f^{(r)}\|_{X_{2\pi}}$, then the corresponding $\mathscr{K}$-functional may be characterized in terms of the rth modulus of continuity

$$\omega_r(t, f; X_{2\pi}) := \sup_{|h| \leqslant t} \left\| \sum_{j=0}^{r} \binom{r}{j} f(\circ + jh) \right\|_{X_{2\pi}} \qquad (f \in X_{2\pi}, t \geqslant 0),$$

i.e., there are constants $c_1, c_2 > 0$ such that (cf. [4, p. 192])

$$c_1 \omega_r(t, f; X_{2\pi}) \leqslant \mathscr{K}(t^r, f; X_{2\pi}, X^{(r)}_{2\pi}) \leqslant c_2 \omega_r(t, f; X_{2\pi}). \tag{3.1}$$

Concerning the applications to algebraic problems treated in Sections 3.5–3.6, let $C[-1, 1]$ denote the space of functions continuous on the compact interval $[-1, 1]$ with the usual maximum norm $\|\circ\|_C$ and

$$U := \left\{ g \in C[-1, 1]; |g|_U := \sup_{-1<x<1} |(1 - x^2) g^{(2)}(x)| < \infty \right\}. \tag{3.2}$$

Then the intermediate spaces $X_\alpha := X_\omega$ for $\omega(t) = t^\alpha$, $0 < \alpha \leqslant 1$, as defined by (1.1–1.2) may be characterized via

$$\begin{gathered} X_\alpha = \left\{ f \in C[-1, 1]; \sup_{-1+h \leqslant x \leqslant 1-h} |(1 - x^2)^\alpha \Delta_h^2 f(x)| = \mathscr{O}(h^{2\alpha}) \right\}, \\ \Delta_h^2 f(x) := f(x - h) - 2f(x) + f(x + h) \qquad (-1 + h \leqslant x \leqslant 1 - h, h \geqslant 0) \end{gathered} \tag{3.3}$$

(cf. [1; 9]). Moreover, for $0 < \alpha < 1$ (and $n \to \infty$)

$$f \in X_\alpha \Leftrightarrow E_n[f] := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_{C[-1,1]} = \mathscr{O}(n^{-2\alpha}), \tag{3.4}$$

where $\mathscr{P}_n$ denotes the set of algebraic polynomials of degree at most n (see [15]).

3.1. *Fourier Partial Sums*

For the Fourier partial sums

$$(S_n f)(x) := \sum_{|k| \leqslant n} f^\wedge(k) e^{ikx}, \qquad f^\wedge(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-iku} \, du,$$

one has the well known direct estimate

$$\|S_n f - f\|_{X_{2\pi}} = \mathcal{O}(\omega_r(n^{-1}, f; X_{2\pi}) \log n).$$

Theorem 2 then reestablishes the fact that this approximation rate is best possible for $X_{2\pi} = C_{2\pi}$, $L^1_{2\pi}$. Indeed,

COROLLARY 2. *Let $X_{2\pi} = C_{2\pi}$ or $L^1_{2\pi}$. For each ω satisfying* (1.3, 1.8) *there exists a function $f_\omega \in X_{2\pi}$ such that*

$$\omega_r(t, f_\omega; X_{2\pi}) = \mathcal{O}(\omega(t^r)) \qquad (t \to 0+),$$
$$\|S_n f_\omega - f_\omega\|_{X_{2\pi}} \neq o(\omega(n^{-r}) \log n) \qquad (n \to \infty).$$

Proof. Let us check the conditions of Theorem 2 for $X = Y = C_{2\pi}$, $U = C^{(r)}_{2\pi}$, and for the linear bounded operators $R_n = [S_n - I]/\log n$, where I is the identity. Since $\|S_n\|_{[C_{2\pi}, C_{2\pi}]} \geqslant c \log n$, there are elements $f_n \in C_{2\pi}$ with $\|f_n\|_{C_{2\pi}} = 1$ and $\|S_n f_n\|_{C_{2\pi}} \geqslant c' \log n$. Now choose $h_n = D_n f_n$, where $D_n := (1/n) \sum_{k=n+1}^{2n} S_k$ are the standard delayed means of de la Vallée Poussin. Then (cf. [5, p. 108]) $\{h_n\}$ is a sequence of trigonometric polynomials of degree $\{2n\}$ satisfying (2.1–2.2) with $\varphi_n = n^{-r}$ as a consequence of the classical Bernstein inequality. Condition (2.3) is fulfilled since for sufficiently large n

$$\|R_n h_n\|_{C_{2\pi}} = \|S_n D_n f_n - D_n f_n\|_{C_{2\pi}}/\log n$$
$$\geqslant [\|S_n f_n\|_{C_{2\pi}} - C]/\log n \geqslant c'' > 0.$$

Then an application of Theorem 2 in connection with the characterization (3.1) completes the proof.

The present treatment may be compared with the classical one given by Lebesgue [11]; for stronger results see [14a].

3.2. *The Singular Integral of de La Vallée Poussin*

For the singular integral of de La Vallée Poussin

$$(V_n f)(x) := \sum_{|k| \leqslant n} \frac{(n!)^2}{(n-k)!\,(n+k)!} f^\wedge(k)\, e^{ikx}$$

it is known that (cf. [5, p. 113])

$$\|V_n f - f\|_{X_{2\pi}} = \mathcal{O}(\omega_2(n^{-1/2}, f; X_{2\pi})) \qquad (f \in X_{2\pi}, n \to \infty). \quad (3.2.1)$$

COROLLARY 3. *For each* ω *satisfying* (1.3, 1.8) *there exists a function* $f_\omega \in X_{2\pi}$ *such that*

$$\omega_2(t, f_\omega; X_{2\pi}) = \mathcal{O}(\omega(t^2)) \qquad (t \to 0+),$$
$$\| V_n f_\omega - f_\omega \|_{X_{2\pi}} \neq \mathcal{O}(\omega(n^{-1})) \qquad (n \to \infty). \tag{3.2.2}$$

Proof. Consider $X = Y = X_{2\pi}$, $U = X_{2\pi}^{(2)}$, and the linear bounded operators $R_n = V_{n^2} - I$. For the elements $h_n \in U$ one may choose $h_n(x) = e^{inx}$. Obviously they satisfy (2.1–2.2) with $\varphi_n = n^{-2}$. Since

$$\|R_n h_n\|_{X_{2\pi}} = \left| \frac{(n^2!)^2}{(n^2-n)!\,(n^2+n)!} - 1 \right| \|h_n\|_{X_{2\pi}}$$

tends to $|e^{-1} - 1| \neq 0$, one also has (2.3) for n large enough. So Theorem 2 and (3.1) establish (3.2.2), first only for the subsequence $\{n^2\}$, but this already implies (3.2.2) completely.

For $\omega(t) = t^\alpha$, $0 < \alpha < 1$, Corollary 3 may be found in [14, p. 184] using the testfunction $|\sin u|^\alpha$. Moreover, in this case even a Bernstein-type inverse theorem is valid; it states that $\| V_n f - f \| = \mathcal{O}(n^{-\alpha/2})$ ensures f to belong to the corresponding Lipschitz class of order α (cf. [5, p. 114]).

3.3 *Best Approximation in* $X_{2\pi}$

Concerning the error of best approximation of a function $f \in X_{2\pi}$ by trigonometric polynomials t_n of degree at most n (i.e., $t_n \in \Pi_n$) one has that (cf. [12, p. 58])

$$E_n^*(f; X_{2\pi}) := \inf_{t_n \in \Pi_n} \|f - t_n\|_{X_{2\pi}} = \mathcal{O}(\omega_r(n^{-1}, f; X_{2\pi})). \tag{3.3.1}$$

COROLLARY 4. *For each* ω *satisfying* (1.3, 1.8) *there exists a function* $f_\omega \in X_{2\pi}$ *such that*

$$\omega_r(t, f_\omega; X_{2\pi}) = \mathcal{O}(\omega(t^r)) \qquad (t \to 0+),$$
$$E_n^*(f_\omega; X_{2\pi}) \neq o(\omega(n^{-r})) \qquad (n \to \infty).$$

Proof. Let us check the conditions of Theorem 2 for $X = X_{2\pi}$, $Y = \mathbf{R}$ (:= set of reals), $U = X_{2\pi}^{(r)}$, and for the sublinear bounded operators $R_n f = E_n^*(f; X_{2\pi})$. For $h_n(x) = \cos(n+1)x$ one has (2.1–2.2) with $\varphi_n = n^{-r}$. Since $g^\wedge(n+1) \leqslant E_n^*(g; X_{2\pi})$ for each $g \in X_{2\pi}$, one also has $|R_n h_n| \geqslant h_n^\wedge(n+1) = 1/2$. So Theorem 2 and (3.1) prove this corollary.

In case $X_{2\pi} = C_{2\pi}$, $\omega(t) = t^\alpha$, $0 < \alpha < 1$, Corollary 4 is also given in [10, p. 55] via a construction of a testfunction similar to a gliding hump method. But the proof uses very specific features of the maximum norm and the one-dimensional trigonometric system. Note that (3.3.1) is again best possible in the sense of a Bernstein inverse theorem (see, e.g., [17, p. 331 ff.]).

3.4. *Compound Quadrature Formulae*

Consider an (n-fold) compound quadrature formula for the approximate calculation of the integral $\int_{-\pi}^{\pi} f(u)\,du$,

$$Q_n f := \sum_{j=1}^{ns} a_{nj} f(x_{nj}) := \frac{1}{n} \sum_{m=0}^{n-1} \sum_{k=1}^{s} b_k f(-\pi + ((2m+1)\pi + y_k)/n),$$

where $f \in C_{2\pi}$, $-\pi \leqslant y_k < \pi$, and $\sum_{k=1}^{s} b_k = 2\pi$. If $f \in C_{2\pi}^{(r-1)}$ is such that $f^{(r-1)}$ is absolutely continuous, then (see, e.g., [2, p. 168])

$$R_n^Q f := Q_n f - \int_{-\pi}^{\pi} f(u)\,du = \left(\frac{2\pi}{n}\right)^r \int_{-\pi}^{\pi} f^{(r)}(u) k_r(nu)\,du, \qquad (3.4.1)$$

where k_r denotes a certain 2π-periodic bounded kernel. Since for $r \geqslant 1$ each $f \in (L_{2\pi}^p)^{(r)}$ is equal a.e. to an $(r-1)$-times absolutely continuous function with rth derivative in $L_{2\pi}^p$ (cf. [5, p. 363]), one may restrict oneself to continuous representatives of $f \in X_{2\pi}^{(r)}$ if $r \geqslant 1$, which in particular leads to an interpretation of (3.4.1) for $f \in X_{2\pi}^{(r)}$. This implies the Jackson-type inequality

$$|R_n^Q f| \leqslant C_r n^{-r} \|f^{(r)}\|_{X_{2\pi}}, \qquad (3.4.2)$$

valid for any $f \in X_{2\pi}^{(r)}$, $r \geqslant 1$. Since also

$$|R_n^Q f| \leqslant C_0 \|f\|_{C_{2\pi}} \qquad (f \in C_{2\pi}, n \in \mathbf{N}), \qquad (3.4.3)$$

Theorem 1 and (3.1) for $X = C_{2\pi}$, $Y = \mathbf{R}$, $U = C_{2\pi}^{(r)}$, and $\varphi_n = n^{-r}$ deliver

$$|R_n^Q f| = \mathcal{O}(\omega_r(n^{-1}, f; C_{2\pi})) \qquad (f \in C_{2\pi}). \qquad (3.4.4)$$

Moreover, one may apply Theorem 1 to

$$R_n = nR_n^Q, \qquad X = \{f \in X_{2\pi}^{(1)}; f^\wedge(0) = 0\}, \qquad \|f\|_X := \|f'\|_{X_{2\pi}}, \qquad (3.4.5)$$
$$Y = \mathbf{R}, \qquad U = X \cap X_{2\pi}^{(r)} = \{f \in X_{2\pi}^{(r)}; f^\wedge(0) = 0\}, \qquad \varphi_n = n^{-r+1}.$$

Since for $f \in X$

$$\mathscr{K}(t, f'; X_{2\pi}, X_{2\pi}^{(r-1)}) \leqslant \mathscr{K}(t, f; X, U) \leqslant 2\mathscr{K}(t, f'; X_{2\pi}, X_{2\pi}^{(r-1)}),$$

the characterization (3.1) yields

$$|R_n^Q f| = \mathcal{O}(n^{-1} \omega_{r-1}(n^{-1}, f'; X_{2\pi})) \qquad (3.4.6)$$

for any $f \in X$, in fact for any $f \in X_{2\pi}^{(1)}$. With Theorem 2 one obtains that (3.4.4, 3.4.6) are best possible.

COROLLARY 5. *For each* ω *satisfying* (1.3, 1.8) *there exists*

(i) *a function* $f_\omega \in C_{2\pi}$ *such that*

$$\omega_r(t, f_\omega; C_{2\pi}) = \mathcal{O}(\omega(t^r)) \qquad (t \to 0+),$$

$$|R_n^Q f_\omega| \neq o(\omega(n^{-r})) \qquad (n \to \infty),$$

(ii) *a functions* $f_\omega \in X_{2\pi}^{(1)}$ *such that*

$$\omega_{r-1}(t, f'_\omega; X_{2\pi}) = \mathcal{O}(\omega(t^{r-1})) \qquad (t \to 0+), \tag{3.4.7}$$

$$|R_n^Q f_\omega| \neq o(n^{-1}\omega(n^{-r+1})) \qquad (n \to \infty). \tag{3.4.8}$$

Proof. (i): For $X = C_{2\pi}$, $Y = \mathbb{R}$, $U = C_{2\pi}^{(r)}$, and for the linear bounded operators $R_n = R_n^Q$ the conditions of Theorem 2 are satisfied for

$$h_n(x) = h(nx), \qquad h(x) := \prod_{k=1}^{s} \sin^2(x - y_k), \qquad \varphi_n = n^{-r}. \tag{3.4.9}$$

Concerning (2.3), note that $h_n(x_{nj}) = 0$, and so

$$|R_n h_n| = \left| Q_n h_n - \int_{-\pi}^{\pi} h_n(u)\, du \right| = \int_{-\pi}^{\pi} h_n(u)\, du = \int_{-\pi}^{\pi} h(u)\, du > 0.$$

(ii): For X, Y, U, R_n as in (3.4.5) the conditions of Theorem 2 hold true. Indeed, with h as given by (3.4.9) the elements $h_n \in U$,

$$h_n(x) = \tilde{h}(nx)/n, \qquad \tilde{h}(x) := h(x) - h^\wedge(0),$$

satisfy (2.1–2.2) with $\varphi_n = n^{-r+1}$ as well as (2.3) since

$$|R_n h_n| = \left| Q_n(nh_n) - \int_{-\pi}^{\pi} nh_n(u)\, du \right| = |Q_n(nh_n)|$$

$$= \sum_{j=1}^{ns} a_{nj} h^\wedge(0) = 2\pi h^\wedge(0) > 0.$$

So an application of Theorem 2 in connection with the corresponding characterization of the $\mathcal{K}$-functional completes the proof.

Let us mention that this example shows that the case $\omega(t) = t$ has to be excluded in Theorems 2, 3. Indeed, for the reflexive spaces $X_{2\pi} = L_{2\pi}^p$, $1 < p < \infty$, condition (3.4.7) implies $f_\omega^{(r)} \in X_{2\pi}$ (see, e.g., [5, p. 368]). But then (3.4.1) implies (cf. [2, p. 220]) that $|R_n^Q f_\omega| = o(n^{-r})$, in contrast to (3.4.8). Thus Corollary 5(ii) is not true in the saturation case $\omega(t) = t$.

3.5. *Polya Quadrature Formula*

Let $x_{nj} := -\cos(2j-1)\pi/2n$, $1 \leqslant j \leqslant n$, denote the zeros of the Chebyshev polynomial $T_n(x) := \cos(n \arccos x) \in \mathscr{P}_n$. Then the Polya quadrature formula Q_n^P for $f \in C[-1, 1]$ is given by

$$Q_n^P f := \sum_{j=1}^{n} a_{nj} f(x_{nj}), \qquad R_n^P f := Q_n^P f - \int_{-1}^{1} f(u)\, du,$$

where the weights $a_{nj} \geqslant 0$ are such that $R_n^P p_{n-1} = 0$ for all $p_{n-1} \in \mathscr{P}_{n-1}$ (cf. [2, p. 116, 136 ff.]). Since for any $p_{n-1} \in \mathscr{P}_{n-1}$

$$|R_n^P f| = |R_n^P(f - p_{n-1})| \leqslant 4 \|f - p_{n-1}\|_{C[-1,1]},$$

one obtains by (3.4) that $f \in X_\alpha$ implies $|R_n^P f| = \mathcal{O}(n^{-2\alpha})$ for each $0 < \alpha < 1$. This result is best possible. Indeed,

COROLLARY 6. *For each* $0 < \alpha < 1$ *there exists* $f_\alpha \in X_\alpha$ *(cf.* (3.3)) *such that* $|R_n^P f_\alpha| \neq o(n^{-2\alpha})$.

Proof. For $X = C[-1, 1]$, $Y = \mathbb{R}$, U as given by (3.2), and the operators $R_n = R_n^P$ the elements $h_n = T_n^2$ satisfy (2.1–2.2) with $\varphi_n = n^{-2}$. Condition (2.3) holds since $h_n(x_{nj}) = 0$, and therefore

$$|R_n h_n| = \left| Q_n^P T_n^2 - \int_{-1}^{1} T_n^2(u)\, du \right| = \int_{-1}^{1} T_n^2(u)\, du$$
$$= 1 - \frac{1}{4n^2 - 1} \geqslant \frac{2}{3}.$$

3.6. *Lagrange Interpolation*

The Lagrange interpolation polynomial of degree $n-1$ with respect to the Chebyshev nodes x_{nj} (cf. Section 3.5) is given by

$$L_{n-1}(f; x) := \sum_{j=1}^{n} 1_{nj}(x) f(x_{nj}), \qquad 1_{nj}(x) := \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - x_{nk}}{x_{nj} - x_{nk}},$$

where $-1 \leqslant x \leqslant 1$ and $f \in C[-1, 1] (=: C)$. Since

$$\|L_n f - f\|_C \leqslant (\|L_n\|_{[C,C]} + 1)\, E_n[f],$$
$$c_1 \log n \leqslant \|L_n\|_{[C,C]} \leqslant c_2 \log n,$$

one obtains by (3.4) that for $0 < \alpha < 1$

$$f \in X_\alpha \Rightarrow \|L_n f - f\|_C = \mathcal{O}(n^{-2\alpha} \log n).$$

On the other hand, an application of Theorem 2 yields

COROLLARY 7. *For each* $0 < \alpha < 1$ *there exists* $f_\alpha \in X_\alpha$ (*cf.* (3.3)) *such that*

$$\|L_n f_\alpha - f_\alpha\| \neq o(n^{-2\alpha} \log n) \qquad (n \to \infty).$$

Proof. The proof is an algebraic version of that of Corollary 2, the delayed means of de la Vallée Poussin being replaced by the Fejér–Hermite operators. Indeed, with $X = Y = C[-1, 1]$, U as given by (3.2), $R_n = [L_n - I]/\log n$, one may consider the elements $h_n = H_n f_n$, where $f_n \in C[-1, 1]$ is such that $\|f_n\|_C = 1$ and $\|L_n f_n\|_C \geqslant c \log n$, and

$$H_n(f; x) := n^{-2} T_n^2(x) \sum_{j=1}^{n} f(x_{nj}) \frac{1 - x x_{nj}}{(x - x_{nj})^2}$$

are the Fejér–Hermite interpolating polynomials of degree $2n-1$ (cf. [14, p. 397 ff.]). Thus (2.1–2.2) are satisfied with $\varphi_n = n^{-2}$ in view of the algebraic Bernstein inequality $|p_n|_U \leqslant cn^2 \|p_n\|_C$ (cf. [17, p. 227]). Concerning (2.3) one has $L_n H_n f = L_n f$ and thus again

$$\|R_n h_n\|_C = \|L_n H_n f_n - H_n f_n\|_C / \log n \geqslant [\|L_n f_n\|_C - c]/\log n$$
$$\geqslant c' > 0.$$

Hence an application of Theorem 2 completes the proof.

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